PLURIHARMONIC MAPPINGS AND LINEARLY CONNECTED DOMAINS IN $\mathbb{C}^n$

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Abstract. In this paper we obtain certain sufficient conditions for the univalence of pluriharmonic mappings defined in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. The results are generalizations of conditions of Chuaqui and Hernández that relate the univalence of planar harmonic mappings with linearly connected domains, and show how such domains can play a role in questions regarding injectivity in higher dimensions. In addition, we extend recent work of Hernández and Martín on a shear type construction for planar harmonic mappings, by adapting the concept of stable univalence to pluriharmonic mappings of the unit ball $\mathbb{B}^n$ into $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by $\mathbb{B}^n_r$ and the unit ball $\mathbb{B}^n_1$ is denoted by $\mathbb{B}^n$. In the case of one complex variable, $\mathbb{B}^1$ is the usual unit disc $\mathbb{U}$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. The space $L(\mathbb{C}^n, \mathbb{C}^n)$ is denoted by $L(\mathbb{C}^n)$. Also, let I_n be the identity in $L(\mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If Ω is a domain in $\mathbb{C}^n$ which contains the origin and $f \in H(\Omega)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. The family of normalized biholomorphic mappings on $\mathbb{B}^n$ will be denoted by $S(\mathbb{B}^n)$. In the case $n = 1$, $S(\mathbb{B}^1)$ is denoted by S, which is the usual family of normalized univalent functions on $\mathbb{U}$. If $f \in H(\mathbb{B}^n)$, we say that f is locally biholomorphic on $\mathbb{B}^n$ if $\det Df(z) \neq 0$, $z \in \mathbb{B}^n$, where $Df(z)$ is the complex Jacobian matrix of f at z. Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on $\mathbb{B}^n$.

A complex-valued function f of class C^2 on $\mathbb{B}^n$ is said to be pluriharmonic if its restriction to every complex line is harmonic, which is equivalent to the fact that

$$\frac{\partial^2}{\partial z_j \partial \overline{z}_k} f(z) = 0, \quad \forall\, z \in \mathbb{B}^n, \quad \forall\, j, k = 1, 2 \dots, n.$$

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Every pluriharmonic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ can be written as $f = h + \overline{g}$, where $g, h \in H(\mathbb{B}^n)$, and this representation is unique if $g(0) = 0$.

If $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ is a pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$, we denote by J_f the real Jacobian of f and $\omega_f(z) = Dg(z)[Dh(z)]^{-1}$ for $z \in \mathbb{B}^n$. Then

$$J_f(z) = \det \begin{pmatrix} Dh(z) & \overline{Dg(z)} \\ Dg(z) & \overline{Dh(z)} \end{pmatrix}, \quad z \in \mathbb{B}^n,$$

and it is elementary to deduce that

$$J_f(z) = |\det Dh(z)|^2 \det(I_n - \omega_f(z)\overline{\omega_f(z)}), \quad z \in \mathbb{B}^n.$$

Hence f is sense-preserving, i.e., $J_f(z) > 0$ for $z \in \mathbb{B}^n$, if and only if h is locally biholomorphic on $\mathbb{B}^n$ and $\det(I_n - \omega_f(z)\overline{\omega_f(z)}) > 0$, for all $z \in \mathbb{B}^n$. In the case of one complex variable, $\omega_f = g'/h'$ is the dilatation of f. It is known that $f = h + \overline{g}$ is locally univalent and sense-preserving on $\mathbb{U}$ if and only if $|g'(z)| < |h'(z)|$ for $z \in \mathbb{U}$, i.e., h is locally univalent on $\mathbb{U}$ and $|\omega_f(z)| < 1$ for $z \in \mathbb{U}$. In dimension $n \geq 2$, if $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ is a pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$ and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$, then f is a sense-preserving locally univalent mapping on $\mathbb{B}^n$ (cf. [6, Theorem 5]).

The following notion will be useful in the next section (see e.g. [11], for $n = 1$).

Definition 1.1. A domain $\Omega \subseteq \mathbb{C}^n$ is called linearly connected if there is a constant $M > 0$ such that any two points $\omega_1, \omega_2 \in \Omega$ can be connected by a smooth curve $\gamma \subset \Omega$ with length $\ell(\gamma) \leq M\|\omega_1 - \omega_2\|$.

Remark 1.2. It is clear that $M \geq 1$ in Definition 1.1 and that any convex domain is linearly connected with constant $M = 1$. On the other hand, if $\Omega_j \subseteq \mathbb{C}$ is a linearly connected domain with constant $M_j > 0$, then it is easy to see that $\Omega = \prod_{j=1}^n \Omega_j$ is a linearly connected domain in $\mathbb{C}^n$ with constant $M = \sqrt{n} \max_{j=1,\dots,n} M_j$.

In the case of one complex variable, every bounded linearly connected domain Ω is a Jordan domain (see [11]). Chuaqui and Hernández [3] proved that if $h \in H(\mathbb{U})$ is a univalent function, then there exists a constant $c > 0$ such that each harmonic function $f = h + \overline{g}$ with $|\omega_f| < c$ is univalent on $\mathbb{U}$ if and only if $h(\mathbb{U})$ is a linearly connected domain.

In this paper, we investigate linear connectivity and its role in the study of certain sufficient conditions of univalence for pluriharmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$, thereby finding n-dimensional analogues of the results in [3]. Other necessary and sufficient conditions of univalence for pluriharmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$ may be found in [6]. On the other hand, Hernández and Martín [8] obtained certain necessary and sufficient conditions for harmonic mappings of the unit disc $\mathbb{U}$ into $\mathbb{C}$ to be stable univalent. We generalize some of these results to the case of pluriharmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$. To this end, we prove that there is an equivalence between stable pluriharmonic univalence and stable analytic univalence on $\mathbb{B}^n$. Also, we prove the equivalence between stable pluriharmonic strongly close-to-convexity and stable analytic close-to-convexity. Other necessary and sufficient conditions of univalence for harmonic and pluriharmonic mappings may be found in [2] and [6].

2. Main results

We begin this section with the following result. In the case of one complex variable, see [3] (see also [1], for related results in the case $n = 1$).

Theorem 2.1. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a pluriharmonic mapping such that h is biholomorphic on $\mathbb{B}^n$ and $h(\mathbb{B}^n)$ is a linearly connected domain with constant $M \geq 1$. Assume that $\|\omega_f(z)\| < 1/M$ for $z \in \mathbb{B}^n$. Then f is univalent and sense-preserving on B^n. Moreover, if $\|\omega_f(z)\| \leq c < 1/M$ for $z \in \mathbb{B}^n$, then $f(\mathbb{B}^n)$ is a linearly connected domain in $\mathbb{C}^n$.*

Proof. Suppose that there exists two distinct points $z_1, z_2 \in \mathbb{B}^n$ such that $f(z_1) = f(z_2)$, or equivalently

$$0 = f(z_1) - f(z_2) = h(z_1) - h(z_2) + \overline{(g(z_1) - g(z_2))} = w_1 - w_2 + \overline{\varphi(w_1) - \varphi(w_2)},$$

where $w_j = h(z_j)$ for $j = 1, 2$, and $\varphi = g \circ h^{-1}$. This implies that

$$\varphi(w_1) - \varphi(w_2) = \overline{w_2 - w_1}. \tag{2.1}$$

Clearly, $w_1 \neq w_2$, since h is injective on $\mathbb{B}^n$. Let $\Gamma \subset h(\mathbb{B}^n)$ be a smooth curve joining w_1 and w_2 such that $\ell(\Gamma) \leq M\|w_1 - w_2\|$. Then, we have

$$\|\varphi(w_1) - \varphi(w_2)\| = \left\| \int_0^1 D\varphi(w(t))(w'(t))dt \right\| \leq \int_0^1 \|D\varphi(w(t))\| \cdot \|w'(t)\|dt, \tag{2.2}$$

where $w(t)$, $0 \leq t \leq 1$, is a parametrization of Γ. On the other hand, since $\varphi = g \circ h^{-1}$, it follows that

$$D\varphi(w) = Dg(z)[Dh(z)]^{-1} = \omega_f(z), \quad z = h^{-1}(w) \in \mathbb{B}^n.$$

Hence, in view of (2.2) and the fact that $\|\omega_f(z)\| < 1/M$ for $z \in \mathbb{B}^n$, we deduce that

$$\|\varphi(w_1) - \varphi(w_2)\| < \frac{1}{M} \int_0^1 \|w'(t)\|dt = \frac{1}{M}\ell(\Gamma) \leq \|w_1 - w_2\|.$$

However, this is a contradiction to (2.1). Hence, f must be univalent, as desired.

Next, assume that $\|\omega_f(z)\| \leq c < 1/M$ for $z \in \mathbb{B}^n$. Let $\Delta = h(\mathbb{B}^n)$ and $\Omega = f(\mathbb{B}^n)$. Also, let $\psi(w) = w + \overline{\varphi(w)}$ for $w \in \Delta$, where $\varphi = g \circ h^{-1}$. Then it is easy to see that $\psi(w) = f(z)$ for $w = h(z) \in \Delta$, and thus $\psi(\Delta) = \Omega$. Now, let ω_1, ω_2 be two distinct points in Ω. Then $\omega_j = \psi(w_j)$, where $w_j \in \Delta$, $j = 1, 2$. Since Δ is linearly connected with constant M, there exists a smooth curve γ contained in Δ such that $\ell(\gamma) \leq M\|w_1 - w_2\|$. Also, let $\Gamma = \psi(\gamma)$. Then Γ is also a smooth curve in Ω between ω_1 and ω_2. We prove that

$$\ell(\Gamma) \leq \frac{(1+c)M}{1-cM}\|\omega_1 - \omega_2\|. \tag{2.3}$$

Since $D_w\psi(w) = I_n$ and $D_{\overline{w}}\psi(w) = \overline{\omega_f(z)}$ for $w = h(z) \in h(\mathbb{B}^n)$, we obtain that

$$\ell(\Gamma) = \int_\Gamma \|du\| = \int_\gamma \|d\psi(w)\| = \int_\gamma \|D_w\psi(w)dw + D_{\overline{w}}\psi(w)d\overline{w}\|$$

$$\leq \int_\gamma (\|I_n\| + \|\omega_f(z)\|)\|dw\| \leq (1+c)\int_\gamma \|dw\| = (1+c)\ell(\gamma).$$

Since $\ell(\gamma) \leq M\|w_1 - w_2\|$, we obtain that

$$\ell(\Gamma) \leq M(1+c)\|w_1 - w_2\|. \tag{2.4}$$

On the other hand, using the fact that

$$\omega_1-\omega_2=w_1-w_2+\overline{\varphi(w_1)-\varphi(w_2)},$$

we deduce that

$$\begin{aligned}\|\omega_1-\omega_2\| &\geq \|w_1-w_2\|-\int_\gamma \|D\varphi(w)dw\|\\ &\geq \|w_1-w_2\|-\int_\gamma \|\omega_f(z)\|\|dw\| \geq \|w_1-w_2\|-c\int_\gamma \|dw\|\\ &= \|w_1-w_2\|-c\ell(\gamma)\geq (1-cM)\|w_1-w_2\|.\end{aligned}$$

Finally, in view of the above relation and (2.4), we obtain that

$$\ell(\Gamma)\leq M(1+c)\|w_1-w_2\|\leq \frac{M(1+c)}{1-cM}\|\omega_1-\omega_2\|.$$

Hence, the relation (2.3) follows, as desired. This completes the proof. □

In view of Theorem 2.1, we obtain the following result (see [6, Theorem 6]). In the case of one complex variable, this result was obtained in [10], [4] and [3].

Corollary 2.2. *Let $h:\mathbb{B}^n\to\mathbb{C}^n$ be a convex (biholomorphic) mapping, and let $f=h+\overline{g}$ be a pluriharmonic mapping such that $\|\omega_f(z)\|<1$ for $z\in\mathbb{B}^n$. Then f is a sense-preserving univalent mapping on $\mathbb{B}^n$. Moreover, if $\|\omega_f(z)\|\leq c<1$ for $z\in\mathbb{B}^n$, then $f(\mathbb{B}^n)$ is a linearly connected domain in $\mathbb{C}^n$.*

The following result provides a sufficient condition of univalence for the analytic part of a pluriharmonic mapping on $\mathbb{B}^n$ whose image is a linearly connected domain (see [3], in the case $n=1$).

Theorem 2.3. *Let $f=h+\overline{g}:\mathbb{B}^n\to\mathbb{C}^n$ be a univalent pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$. Assume that $f(\mathbb{B}^n)$ is a linearly connected domain in $\mathbb{C}^n$ with constant $M\geq 1$, and $\|\omega_f(z)\|<1/(1+M)$ for $z\in\mathbb{B}^n$. Then h is biholomorphic on $\mathbb{B}^n$.*

Proof. First, we observe that f is a sense-preserving mapping, since $\|\omega_f(z)\|<1/(1+M)<1$ for $z\in\mathbb{B}^n$. Suppose that there exist two distinct points $z_1,z_2\in\mathbb{B}^n$ such that $h(z_1)=h(z_2)$. Then $f(z_1)-f(z_2)=\overline{g(z_1)-g(z_2)}$, i.e.

$$w_1-w_2=\overline{\varphi(w_1)-\varphi(w_2)}, \tag{2.5}$$

where $w_j=f(z_j)$ and $\varphi=g\circ f^{-1}$. Clearly, $w_1\neq w_2$, and since $f(\mathbb{B}^n)$ is a linearly connected domain with constant M, there exists a smooth curve $\Gamma\subset f(\mathbb{B}^n)$ between w_1 and w_2 such that $\ell(\Gamma)\leq M\|w_1-w_2\|$. In view of (2.5) and the above relation, we obtain that

$$\|w_1-w_2\|=\|\varphi(w_1)-\varphi(w_2)\|=\left\|\int_\Gamma D_w\varphi(w)dw+D_{\overline{w}}\varphi(w)d\overline{w}\right\|. \tag{2.6}$$

It is easy to see that

$$D_w\varphi(w)=Dg(z)D_wf^{-1}(w)\ \text{ and }\ D_{\overline{w}}\varphi(w)=Dg(z)D_{\overline{w}}f^{-1}(w),$$

for all $w=f(z)\in f(\mathbb{B}^n)$. Also, since $(f^{-1}\circ f)(z)=z$, it follows that

$$\begin{aligned}D_wf^{-1}(w)Dh(z)+D_{\overline{w}}f^{-1}(w)Dg(z) &= I_n\\ D_wf^{-1}(w)\overline{Dg(z)}+D_{\overline{w}}f^{-1}(w)\overline{Dh(z)} &= \mathbf{0}_n.\end{aligned}$$

Since $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$, it follows that $I_n - \omega_f(z)\overline{\omega_f(z)}$ is an invertible operator. In view of the above relations, we deduce that

$$\begin{aligned} D_w f^{-1}(w) &= [Dh(z)]^{-1}\left(I_n - \overline{Dg(z)[Dh(z)]^{-1}}Dg(z)[Dh(z)]^{-1}\right)^{-1} \\ &= [Dh(z)]^{-1}\left(I_n - \overline{\omega_f(z)}\omega_f(z)\right)^{-1}, \quad w = f(z) \in f(\mathbb{B}^n), \end{aligned}$$

and

$$D_{\overline{w}} f^{-1}(w)$$

$$\begin{aligned} &= -[Dh(z)]^{-1}\left(I_n - \overline{Dg(z)[Dh(z)]^{-1}}Dg(z)[Dh(z)]^{-1}\right)^{-1}\overline{Dg(z)[Dh(z)]^{-1}} \\ &= -[Dh(z)]^{-1}\left(I_n - \overline{\omega_f(z)}\omega_f(z)\right)^{-1}\overline{\omega_f(z)}, \quad w = f(z) \in f(\mathbb{B}^n). \end{aligned}$$

Taking into account the above relations, we deduce that

$$\|\varphi(w_1) - \varphi(w_2)\| \leq \tag{2.7}$$

$$\begin{aligned} &\leq \int_\Gamma \|Dg(f^{-1}(w))D_w f^{-1}(w)dw + Dg(f^{-1}(w))D_{\overline{w}} f^{-1}(w)d\overline{w}\| \\ &= \int_\Gamma \|\omega_f(f^{-1}(w))(I_n - \overline{\omega_f(f^{-1}(w))}\omega_f(f^{-1}(w)))^{-1}(I_n dw - \overline{\omega_f(f^{-1}(w))})d\overline{w})\| \\ &\leq \int_\Gamma \frac{\|\omega_f(f^{-1}(w))\|}{1 - \|\omega_f(f^{-1}(w))\|^2}(1 + \|\omega_f(f^{-1}(w))\|)\|dw\| = \int_\Gamma \frac{\|\omega_f(f^{-1}(w))\| \cdot \|dw\|}{1 - \|\omega_f(f^{-1}(w))\|} \\ &< \frac{1/(1+M)}{1 - 1/(1+M)}\int_\Gamma \|dw\| = \frac{1}{M}\ell(\Gamma) \leq \|w_1 - w_2\|. \end{aligned}$$

However, this is a contradiction to (2.6). Hence, h must be univalent, as desired. This completes the proof. □

In view of Theorem 2.3, we deduce the following particular case. This result is an n-dimensional version of [3, Theorem 2].

Corollary 2.4. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$. Assume that $f(\mathbb{B}^n)$ is a convex domain in $\mathbb{C}^n$ and $\|\omega_f(z)\| < 1/2$ for $z \in \mathbb{B}^n$. Then h is biholomorphic on $\mathbb{B}^n$.*

We next prove that under the assumptions of Theorem 2.3, if $\|\omega_f(z)\| \leq c$, $z \in \mathbb{B}^n$, for some constant $c < 1/(1+M)$, then $h(\mathbb{B}^n)$ is a linearly connected domain (see [3], in the case $n = 1$). We have

Theorem 2.5. *Let $f = h+\overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$. Assume that $f(\mathbb{B}^n)$ is a linearly connected domain with constant $M \geq 1$ and $\|\omega_f(z)\| \leq c$ for $z \in \mathbb{B}^n$, where $c < 1/(1+M)$. Then h maps $\mathbb{B}^n$ onto a linearly connected domain in $\mathbb{C}^n$.*

Proof. In view of Theorem 2.3, we deduce that h is biholomorphic on $\mathbb{B}^n$. Let $\Delta = h(\mathbb{B}^n)$ and $\Omega = f(\mathbb{B}^n)$. Also, let $\psi(w) = w - \overline{\varphi(w)}$ for $w \in \Omega$, where $\varphi = g \circ f^{-1}$. Then it is easy to see that $\psi(w) = h(z)$ for $w = f(z) \in \Omega$, and thus $\psi(\Omega) = \Delta$. Now, let ω_1, ω_2 be two distinct points in Δ. Then $\omega_j = \psi(w_j)$, where $w_j \in \Omega$, $j = 1, 2$. Since Ω is linearly connected with constant M, there exists a smooth curve γ contained in Ω such that $\ell(\gamma) \leq M\|w_1 - w_2\|$. Also, let $\Gamma = \psi(\gamma)$. Then Γ is also a smooth curve in Δ between ω_1 and ω_2. We prove that

$$\ell(\Gamma) \leq \frac{M}{1 - c(1+M)}\|\omega_1 - \omega_2\|. \tag{2.8}$$

To this end, we use arguments similar to those in the proof of Theorem 2.3, to deduce the following relations

$$\begin{aligned} D_w\psi(w) &= I_n + \\ &+ \overline{Dg(z)[Dh(z)]^{-1}}\left(I_n - Dg(z)[Dh(z)]^{-1}\overline{Dg(z)[Dh(z)]^{-1}}\right)^{-1} Dg(z)[Dh(z)]^{-1} \\ &= I_n + \overline{\omega_f(z)}\left(I_n - \omega_f(z)\overline{\omega_f(z)}\right)^{-1}\omega_f(z), \quad w = f(z) \in \Omega, \end{aligned}$$

and

$$\begin{aligned} D_{\overline{w}}\psi(w) &= -D_{\overline{w}}\overline{\varphi(w)} \\ &= -\overline{Dg(z)[Dh(z)]^{-1}}\left(I_n - Dg(z)[Dh(z)]^{-1}\overline{Dg(z)[Dh(z)]^{-1}}\right)^{-1} \\ &= -\overline{\omega_f(z)}\left(I_n - \omega_f(z)\overline{\omega_f(z)}\right)^{-1}, \quad w = f(z) \in f(\mathbb{B}^n). \end{aligned}$$

In view of the above relations, we obtain that

$$\begin{aligned} \ell(\Gamma) &= \int_\Gamma \|du\| = \int_\gamma \|d\psi(w)\| = \int_\gamma \|D_w\psi(w)dw + D_{\overline{w}}\psi(w)d\overline{w}\| \\ &\leq \int_\gamma \|dw\| + \int_\gamma \|\omega_f(z)\|\frac{\|dw\|}{1 - \|\omega_f(z)\|} \leq \frac{1}{1-c}\int_\gamma \|dw\| = \frac{1}{1-c}\ell(\gamma). \end{aligned}$$

Since $\ell(\gamma) \leq M\|w_1 - w_2\|$, we obtain that

$$\ell(\Gamma) \leq \frac{M}{1-c}\|w_1 - w_2\|. \tag{2.9}$$

On the other hand, using the fact that

$$\omega_1 - \omega_2 = w_1 - w_2 - \overline{\varphi(w_1) - \varphi(w_2)},$$

we deduce that

$$\begin{aligned} \|\omega_1 - \omega_2\| &\geq \|w_1 - w_2\| - \int_\gamma \|d\varphi(w)\| \\ &\geq \|w_1 - w_2\| - \int_\gamma \|D_w\varphi(w)dw + D_{\overline{w}}\varphi(w)d\overline{w}\| \\ &\geq \|w_1 - w_2\| - \int_\gamma \frac{\|\omega_f(z)\|}{1 - \|\omega_f(z)\|}\|dw\| \geq \|w_1 - w_2\| - \frac{c}{1-c}\int_\gamma \|dw\| \\ &= \|w_1 - w_2\| - \frac{c}{1-c}\ell(\gamma) \geq \frac{1 - c(1+M)}{1-c}\|w_1 - w_2\|. \end{aligned}$$

Finally, in view of the above relation and (2.9), we obtain that

$$\ell(\Gamma) \leq \frac{M}{1-c}\|w_1 - w_2\| \leq \frac{M}{1 - c(1+M)}\|\omega_1 - \omega_2\|.$$

Hence, the relation (2.8) follows, as desired. This completes the proof. □

In view of Theorem 2.5, we obtain the following particular case.

Corollary 2.6. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$. Assume that $f(\mathbb{B}^n)$ is a convex domain in $\mathbb{C}^n$ and $\|\omega_f(z)\| \le c$ for $z \in \mathbb{B}^n$, where $c < 1/2$. Then h maps $\mathbb{B}^n$ onto a linearly connected domain in $\mathbb{C}^n$.*

Remark 2.7. Let $f = h+\overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a sense-preserving univalent pluriharmonic mapping such that h is locally biholomorphic on $\mathbb{B}^n$. Assume that $f(\mathbb{B}^n)$ is a convex domain in $\mathbb{C}^n$. It would be interesting to see if h is biholomorphic on $\mathbb{B}^n$. In the case of one complex variable, this property is true in view of [9, Theorem 2.1].

The following result provides a sufficient condition for a pluriharmonic mapping f of $\mathbb{B}^n$ onto a linearly connected domain to be stable univalent in the sense of Definition 3.1. This result is a generalization of [3, Theorem 3].

Theorem 2.8. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent pluriharmonic mapping. Assume that $f(\mathbb{B}^n)$ is a linearly connected domain with constant $M \ge 1$. If $\|\omega_f(z)\| < 1/(1 + 2M)$ for $z \in \mathbb{B}^n$, then $f_A = h + A\overline{g}$ is univalent and sense-preserving on $\mathbb{B}^n$, for each $A \in L(\mathbb{C}^n)$ with $\|A\| \le 1$. Moreover, if $\|\omega_f(z)\| \le c < 1/(1 + 2M)$ for $z \in \mathbb{B}^n$, then $f_A(\mathbb{B}^n)$ is a linearly connected domain in $\mathbb{C}^n$.*

Proof. Fix $A \in L(\mathbb{C}^n)$ such that $\|A\| \le 1$ and $A \ne I_n$. Since $\|\omega_{f_A}(z)\| \le \|\omega_f(z)\| < 1/(1 + 2M)$ for $z \in \mathbb{B}^n$, we deduce that f and f_A are sense-preserving mappings. Suppose that there exist two distinct points $z_1, z_2 \in \mathbb{B}^n$ such that $f_A(z_1) = f_A(z_2)$. This relation implies that

$$f(z_1) - f(z_2) = (I_n - A)\overline{(g(z_1) - g(z_2))}.$$

Let $w_1 = f(z_1)$ and $w_2 = f(z_2)$. Then

$$w_1 - w_2 = (I_n - A)\overline{(\varphi(w_1) - \varphi(w_2))}, \quad \varphi = g \circ f^{-1}.$$

As in the proof of (2.7), we deduce that

$$\|w_1 - w_2\| \le \frac{C}{1 - C} 2M\|w_1 - w_2\|, \tag{2.10}$$

where $C = \sup_{z\in f^{-1}(\Gamma)} \|\omega_f(z)\|$. On the other hand, since $C < 1/(1 + 2M)$, the relation (2.10) holds if and only if $w_1 = w_2$, which implies that $z_1 = z_2$. However, this is a contradiction. Hence f_A is univalent on $\mathbb{B}^n$, as desired.

Next, we assume that $\|\omega_f(z)\| \le c < 1/(1 + 2M)$ for $z \in \mathbb{B}^n$. By Theorem 2.5 and its proof, $h(\mathbb{B}^n)$ is a linearly connected domain with constant

$$\frac{M}{1 - (1 + M)/(1 + 2M)} = 1 + 2M.$$

By applying Theorem 2.1 to the mapping f_A, we obtain that $f_A(\mathbb{B}^n)$ is a linearly connected domain in $\mathbb{C}^n$. This completes the proof. □

Corollary 2.9. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent pluriharmonic mapping. Assume that $f(\mathbb{B}^n)$ is a convex domain. If $\|\omega_f(z)\| < 1/3$ for $z \in \mathbb{B}^n$, then $f_A = h + A\overline{g}$ is univalent and sense-preserving on $\mathbb{B}^n$, for each $A \in L(\mathbb{C}^n)$ with $\|A\| \le 1$. Moreover, if $\|\omega_f(z)\| \le c < 1/3$, $z \in \mathbb{B}^n$, then $f_A(\mathbb{B}^n)$ is a linearly connected domain in $\mathbb{C}^n$.*

Before to give the following remarks, we recall that if $f = h + \overline{g} : \mathbb{U} \to \mathbb{C}$ is a harmonic mapping, then f is called close-to-convex if f is univalent and $f(\mathbb{U})$ is a close-to-convex domain, i.e., $\mathbb{C} \setminus f(\mathbb{U})$ is a union of non-crossing half-lines. It is well known in the analytic case that f is close-to-convex on $\mathbb{U}$ if and only if there exists a convex (univalent) function h such that

$$\Re\left[\frac{f'(z)}{h'(z)}\right] > 0, \quad z \in \mathbb{U}.$$

Remark 2.10. Kalaj [9] (compare [3, Theorem 3]) proved that if $f = h + \overline{g}$ is a sense-preserving univalent harmonic mapping of the unit disc $\mathbb{U}$ onto a convex domain, then $f_a = h + a\overline{g}$ is close-to-convex, and thus univalent, for all $a \in \mathbb{C}$, $|a| \leq 1$. In addition, if $|a| < 1$, then f_a is $|a|$-quasiconformal.

Remark 2.11. Let $n \geq 2$ and let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a univalent pluriharmonic mapping. Assume that $f(\mathbb{B}^n)$ is a convex domain and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$. It would be interesting to see if $f_A = h + A\overline{g}$ is univalent on $\mathbb{B}^n$, for all $A \in L(\mathbb{C}^n)$ such that $\|A\| \leq 1$, and if f_A is quasiconformal on $\mathbb{B}^n$, whenever $\|A\| < 1$, respectively.

3. Stable univalent mappings on $\mathbb{B}^n$

In this section we investigate the connection between stable pluriharmonic univalent mappings and stable analytic univalent mappings. In the case of one complex variable, this notion was considered in [8].

Definition 3.1. Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a sense preserving pluriharmonic mapping. We say that f is stable univalent on $\mathbb{B}^n$ if all mappings $f_A = h + A\overline{g}$, where A is a unitary matrix, are univalent on $\mathbb{B}^n$.

We also say that the analytic mapping $h + g$ is stable univalent on $\mathbb{B}^n$ if all mappings $F_A = h + Ag$, where A is a unitary matrix, are univalent on $\mathbb{B}^n$.

Theorem 3.2. *The sense preserving pluriharmonic mapping $f = h + \overline{g}$ is stable pluriharmonic univalent on $\mathbb{B}^n$ if and only if the analytic mapping $F = h + g$ is stable analytic univalent on $\mathbb{B}^n$.*

Proof. Assume that $f = h + \overline{g}$ is stable pluriharmonic univalent on $\mathbb{B}^n$. If $F = h + g$ is not stable analytic univalent on $\mathbb{B}^n$, then there exists a unitary matrix A such that $F_A = h + Ag$ is not univalent on $\mathbb{B}^n$. Then there exist distinct points $z_1, z_2 \in \mathbb{B}^n$ such that $F_A(z_1) = F_A(z_2)$. Therefore, we have

$$h(z_1) - h(z_2) = A(g(z_2) - g(z_1)).$$

If $h(z_1) = h(z_2)$, then we have $g(z_1) = g(z_2)$, and this implies that f is not univalent. Hence $h(z_1) \neq h(z_2)$. Then there exists a unitary matrix V such that

$$V(h(z_1) - h(z_2)) = VA(g(z_2) - g(z_1))$$

is a real vector. Then we have

$$V(h(z_1) - h(z_2)) = \overline{VA}(\overline{g(z_2)} - \overline{g(z_1)}).$$

This implies that $f_{V^{-1}\overline{VA}}(z_1) = f_{V^{-1}\overline{VA}}(z_2)$. However, this is a contradiction to the fact that $f_{V^{-1}\overline{VA}}$ is univalent on $\mathbb{B}^n$. Hence $F = h + g$ is stable analytic univalent on $\mathbb{B}^n$, as desired.

The converse part can be proved by an argument similar to the above. □

In view of Theorem 3.2, we obtain the following sufficient condition for a sense-preserving pluriharmonic mapping to be univalent on $\mathbb{B}^n$ (compare [6]).

Corollary 3.3. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a sense-preserving pluriharmonic mapping. If $h + Ag$ is biholomorphic on $\mathbb{B}^n$, for each unitary matrix A, then f is univalent on $\mathbb{B}^n$.*

Proof. Indeed, since $h + Ag$ is biholomorphic for each unitary matrix A, it follows that $h + g$ is stable analytic univalent on $\mathbb{B}^n$, and thus f is stable univalent on $\mathbb{B}^n$, in view of Theorem 3.2. Hence f is also univalent, as desired. □

From Corollary 2.2 we obtain the following sufficient condition for a pluriharmonic mapping to be stable univalent on $\mathbb{B}^n$.

Corollary 3.4. *Let $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ be a pluriharmonic mapping such that h is convex (biholomorphic) on $\mathbb{B}^n$ and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$. Then f is stable pluriharmonic univalent on $\mathbb{B}^n$.*

Proof. Clearly, f is sense-preserving since $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$. Let A be a unitary matrix, and let $f_A = h + A\overline{g}$. Since $\omega_{f_A}(z) = \overline{A}\omega_f(z)$ for $z \in \mathbb{B}^n$, we deduce that $\|\omega_{f_A}(z)\| < 1$ for $z \in \mathbb{B}^n$. Since h is convex, it follows in view of Corollary 2.2 that f_A is univalent. Also, since A is arbitrary, we deduce that f is stable univalent, as desired. This completes the proof. □

Remark 3.5. Clearly, any stable pluriharmonic univalent mapping on $\mathbb{B}^n$ is also univalent on $\mathbb{B}^n$. However, there exist pluriharmonic univalent mappings on $\mathbb{B}^n$ which are not stable univalent. To see this, let $h, g : \mathbb{U} \to \mathbb{C}$ be given by (cf. [8])

$$h(\zeta) = \frac{\zeta - \frac{1}{2}\zeta^2 + \frac{1}{6}\zeta^3}{(1-\zeta)^3} \text{ and } g(\zeta) = \frac{\frac{1}{2}\zeta^2 + \frac{1}{6}\zeta^3}{(1-\zeta)^3}, \quad |\zeta| < 1.$$

Then

$$h(\zeta) + g(\zeta) = \frac{\zeta + \frac{1}{3}\zeta^3}{(1-\zeta)^3}, \quad |\zeta| < 1.$$

The above relation implies that $|h(r) + g(r)| > \frac{r}{(1-r)^2}$ for $r \in (0, 1)$, and thus $h + g \notin S$. However, the Koebe harmonic function $f = h + \overline{g}$ is univalent on $\mathbb{U}$ (see e.g. [5]). Now, let $H(z) = (h(z_1), \ldots, h(z_n))$ and $G(z) = (g(z_1), \ldots, g(z_n))$ for $z = (z_1, \ldots, z_n) \in \mathbb{B}^n$. Also, let $F = H + \overline{G}$. It is clear that that $H + G$ is not biholomorphic on $\mathbb{B}^n$, in view of the fact that $h + g$ is not univalent on $\mathbb{U}$. Taking into account Theorem 3.2, we deduce that f is not stable univalent on $\mathbb{B}^n$. On the other hand, since $F(z) = (f(z_1), \ldots, f(z_n))$ for $z = (z_1, \ldots, z_n) \in \mathbb{B}^n$, and f is univalent on $\mathbb{U}$, it follows that F is also univalent on $\mathbb{B}^n$.

Next, we prove the following result related to the univalence of the analytic part of a stable pluriharmonic univalent mapping on $\mathbb{B}^n$.

Theorem 3.6. *Let $f = h + \overline{g}$ be a stable pluriharmonic univalent mapping on $\mathbb{B}^n$ such that h is locally biholomorphic on $\mathbb{B}^n$ and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$. Then h is biholomorphic on $\mathbb{B}^n$.*

Proof. Suppose that h is not univalent on $\mathbb{B}^n$. Then there exist two distinct points $z_1, z_2 \in \mathbb{B}^n$ such that $h(z_1) = h(z_2)$. By considering $\tilde{f} = (h \circ \varphi_1 - h(z_1)) + \overline{(g \circ \varphi_1 - g(z_1))}$, where $\varphi_1 \in \mathrm{Aut}(\mathbb{B}^n)$ with $\varphi_1(0) = z_1$, we may assume that $z_1 = h(z_1) = 0$ and $g(z_1) = 0$.

Since $h(0) = g(0) = 0$ and $F_U = h + Ug$ is univalent on $\mathbb{B}^n$ for any unitary matrix U, we obtain that for all $z \in \mathbb{B}^n \setminus \{0\}$, $\|h(z)\| \neq \|g(z)\|$. Using the continuity of $\|h(z)\| - \|g(z)\|$ on $\mathbb{B}^n$ and the assumption that $h(z_2) = h(z_1) = 0$, we have

$\|h(z)\| < \|g(z)\|$ on $\mathbb{B}^n \setminus \{0\}$. Since h is locally biholomorphic, there exists $z_3 \in \mathbb{B}^n \setminus \{0\}$ such that $h(tz_3) \neq 0$ for $t \in (0,1)$. From $\|h(tz_3)\| < \|g(tz_3)\|$ we have $\|Dh(0)z_3 + O(t)\| < \|Dg(0)z_3 + O(t)\|$. Letting $t \to +0$ in this inequality, we have $\|Dh(0)z_3\| \leq \|Dg(0)z_3\|$. Therefore, we have $\|w\| \leq \|\omega_f(0)w\|$, where $w = Dh(0)z_3 \neq 0$. This is a contradiction. Thus, h is univalent on $\mathbb{B}^n$, as desired. This completes the proof. □

We next consider the notion of stable strongly close-to-convexity for pluriharmonic mappings on $\mathbb{B}^n$ and relate this notion to that of holomorphic close-to-convexity.

The following notion is due to Suffridge [12]. Note that any close-to-convex mapping on $\mathbb{B}^n$ is also biholomorphic (see [12] and [7]).

Definition 3.7. Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a holomorphic mapping. We say that f is close-to-convex if there exists a convex (biholomorphic) mapping h on $\mathbb{B}^n$ such that

$$\Re\langle Df(z)[Dh(z)]^{-1}(u), u\rangle > 0, \quad z \in \mathbb{B}^n, \quad \|u\| = 1.$$

The above notion may be extended to the case of mappings of class C^1 on $\mathbb{B}^n$.

Definition 3.8. Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a mapping of class C^1 on $\mathbb{B}^n$. We say that f is strongly close-to-convex if there exists a convex (biholomorphic) mapping h on $\mathbb{B}^n$ such that

$$\Re\langle D_z f(z)[Dh(z)]^{-1}(u) + D_{\overline{z}} f(z)\overline{[Dh(z)]^{-1}}(\overline{u}), u\rangle > 0, \quad z \in \mathbb{B}^n, \quad \|u\| = 1. \tag{3.1}$$

It is clear that a mapping $f \in H(\mathbb{B}^n)$ is strongly close-to-convex if and only if f is close-to-convex in the sense of Definition 3.7.

We next prove that any C^1 strongly close-to-convex mapping on $\mathbb{B}^n$ is univalent (see [4] and [10] in the case $n = 1$).

Proposition 3.9. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a C^1 strongly close-to-convex mapping. Then f is univalent on $\mathbb{B}^n$.*

Proof. Since f is strongly close-to-convex, there exists a convex (biholomorphic) mapping h on $\mathbb{B}^n$ such that the relation (3.1) holds. Let $\Delta = h(\mathbb{B}^n)$ and $q : \Delta \to \mathbb{C}^n$ be given by $q = f \circ h^{-1}$. Then q is of class C^1 on Δ and it is easy to see that

$$\Re\langle D_w q(w)(u) + D_{\overline{w}} q(w)(\overline{u}), u\rangle > 0, \quad w \in \Delta, \quad \|u\| = 1.$$

Now, let w_1 and w_2 be arbitrary points in Δ such that $w_1 \neq w_2$. Then $w(t) = (1-t)w_1 + tw_2 \in \Delta$, $t \in [0,1]$, by the convexity of Δ, and

$$\Re\left\langle q(w_2) - q(w_1), \frac{w_2 - w_1}{\|w_2 - w_1\|^2}\right\rangle = \int_0^1 \Re\left\langle \frac{d}{dt} q(w(t)), \frac{w_2 - w_1}{\|w_2 - w_1\|^2}\right\rangle$$

$$= \int_0^1 \Re\left\langle D_w q(w(t))\left(\frac{w_2 - w_1}{\|w_2 - w_1\|}\right) + D_{\overline{w}} q(w(t))\left(\frac{\overline{w_2} - \overline{w_1}}{\|w_2 - w_1\|}\right), \frac{w_2 - w_1}{\|w_2 - w_1\|}\right\rangle > 0.$$

Consequently, $q(w_1) \neq q(w_2)$, and thus q is univalent on Δ. Hence $f = q \circ h$ is also univalent on $\mathbb{B}^n$, as desired. This completes the proof. □

Remark 3.10. (i) It is not difficult to deduce that if $f = h + \overline{g} : \mathbb{B}^n \to \mathbb{C}^n$ is a pluriharmonic mapping such that h is convex (biholomorphic) on $\mathbb{B}^n$ and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$, then f is strongly-close-to-convex.

Indeed, the condition (3.1) reduces to

$$1 + \Re\langle Dg(z)[Dh(z)]^{-1}(u), \overline{u}\rangle > 0, \quad z \in \mathbb{B}^n, \quad \|u\| = 1.$$

Since $\|\omega_f(z)\| < 1$, the above condition holds, as desired.

(ii) If $f = h + \overline{g} : \mathbb{U} \to \mathbb{C}$ is a sense-preserving harmonic mapping such that h is convex, then f is close-to-convex, in view of [4, Theorem 5.17].

(iii) Let $f = h + \overline{g} : \mathbb{U} \to \mathbb{C}$ be a harmonic mapping such that h is convex. If f is strongly close-to-convex with respect to h, then f is also close-to-convex.

Indeed, the condition (3.1) reduces to

$$1 + \Re\left[u^2 \frac{g'(z)}{h'(z)}\right] > 0, \quad z \in \mathbb{U}, \quad \|u\| = 1.$$

Hence $|\omega_f(z)| < 1$ for $z \in \mathbb{U}$, and thus f is sense-preserving. In view of Remark 3.10 (ii), if follows that f is close-to-convex, as desired.

Now, we may define the notion of stable strongly close-to-convexity for pluriharmonic mappings on $\mathbb{B}^n$ (cf. [8]).

Definition 3.11. Let $f = h + \overline{g}$ be a sense-preserving pluriharmonic mapping on $\mathbb{B}^n$. We say that f is stable pluriharmonic strongly close-to-convex (with respect to a biholomorphic convex mapping H) if all mappings $f_A = h + A\overline{g}$, where A is a unitary matrix, are strongly close-to-convex (with respect to H) on $\mathbb{B}^n$.

We also say that the analytic mapping $h + g$ is stable close-to-convex (with respect to a biholomorphic convex mapping H) on $\mathbb{B}^n$ if all mappings $F_A = h + Ag$, where A is a unitary matrix, are close-to-convex (with respect to H) on $\mathbb{B}^n$.

Theorem 3.12. *Let $f = h + \overline{g}$ be a pluriharmonic univalent mapping on $\mathbb{B}^n$ such that h is convex (biholomorphic) on $\mathbb{B}^n$ and $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$. Then f is stable pluriharmonic strongly close-to-convex. Also, $h + g$ is stable analytic close-to-convex.*

Proof. Let $f_A = h + A\overline{g}$, where A is a unitary matrix. Since $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$, it is easy to deduce that

$$\begin{aligned}&\Re\langle D_z f_A(z)[Dh(z)]^{-1}(w) + D_{\overline{z}} f_A(z)\overline{[Dh(z)]^{-1}}(\overline{w}), w\rangle \\ &= 1 + \Re\langle A\overline{\omega_f(z)(w)}, w\rangle > 0, \quad z \in \mathbb{B}^n, \quad \|w\| = 1.\end{aligned}$$

Since h is convex, it follows that f_A is strongly close-to-convex with respect to h, as desired.

The fact that $h + Ag$ is close-to-convex with respect to h follows in the same manner as above. □

We can prove the converse of Theorem 3.12 (compare [8] for $n = 1$).

Theorem 3.13. *Let $f = h + \overline{g}$ be a pluriharmonic univalent mapping on $\mathbb{B}^n$ such that h is convex (biholomorphic) on $\mathbb{B}^n$. If f is stable pluriharmonic strongly close-to-convex with respect to h or $h + g$ is stable analytic close-to-convex with respect to h, then $\|\omega_f(z)\| < 1$ for $z \in \mathbb{B}^n$.*

Proof. Let $z \in \mathbb{B}^n$ be fixed. There exists $w \in \mathbb{C}^n$ with $\|w\| = 1$ such that $\|\omega_f(z)\| = \|\omega_f(z)(w)\|$. We may assume that $\|\omega_f(z)\| > 0$. Assume that f is stable pluriharmonic strongly close-to-convex with respect to h. Let $f_A = h + A\overline{g}$, where A is a unitary matrix such that

$$-\frac{\overline{\omega_f(z)(w)}}{\|\omega_f(z)(w)\|} = A^* w.$$

Since

$$\begin{aligned}
&\Re\langle D_z f_A(z)[Dh(z)]^{-1}(w) + D_{\overline{z}} f_A(z)\overline{[Dh(z)]^{-1}}(\overline{w}), w\rangle \\
&= 1 + \Re\langle A\overline{\omega_f(z)(w)}, w\rangle = 1 - \|\omega_f(z)(w)\| > 0,
\end{aligned}$$

we obtain that $\|\omega_f(z)\| < 1$, as desired. □

We next prove that there is an equivalence between stable pluriharmonic strongly close-to-convexity and stable analytic close-to-convexity on $\mathbb{B}^n$ (compare [8] for $n = 1$; see also [4] and [5]).

Theorem 3.14. *Let $f = h+\overline{g}$ be a pluriharmonic univalent mapping on $\mathbb{B}^n$. Let H be a biholomorphic convex mapping on $\mathbb{B}^n$. Then f is stable strongly pluriharmonic close-to-convex with respect to H if and only if $F = h + g$ is stable analytic close-to-convex with respect to H.*

Proof. Assume that $f = h+\overline{g}$ is stable pluriharmonic strongly close-to-convex with respect to H. Let $F_A = h+Ag$, where A is an arbitrary unitary matrix. Let $w \in \mathbb{C}^n$ with $\|w\| = 1$ be fixed. There exists a unitary matrix U such that $A^*w = \overline{U^*w}$. Since the pluriharmonic mapping $f_U = h + U\overline{g}$ is strongly close-to-convex with respect to H, we obtain that

$$\Re\langle D_z f_U(z)[DH(z)]^{-1}(w) + D_{\overline{z}} f_U(z)\overline{[DH(z)]^{-1}}(\overline{w}), w\rangle > 0.$$

However, since

$$\begin{aligned}
&\Re\langle D_z f_U(z)[DH(z)]^{-1}(w) + D_{\overline{z}} f_U(z)\overline{[DH(z)]^{-1}}(\overline{w}), w\rangle \\
&= \Re\langle Dh(z)[DH(z)]^{-1}(w), w\rangle + \Re\langle U\overline{Dg(z)[DH(z)]^{-1}}(\overline{w}), w\rangle \\
&= \Re\langle Dh(z)[DH(z)]^{-1}(w), w\rangle + \Re\langle Dg(z)[DH(z)]^{-1}(w), \overline{U^*w}\rangle \\
&= \Re\langle Dh(z)[DH(z)]^{-1}(w), w\rangle + \Re\langle ADg(z)[DH(z)]^{-1}(w), w\rangle \\
&= \Re\langle DF_A(z)[DH(z)]^{-1}(w), w\rangle,
\end{aligned}$$

we obtain that

$$\Re\langle DF_A(z)[DH(z)]^{-1}(w), w\rangle > 0.$$

Thus, F is stable analytic close-to-convex with respect to H.

The converse part can be proved by an argument similar to the above. □

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